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Error of the Local Sound Field's Maximum Frequency Shifts in Shallow Water

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Abstract—The error in determining the position of the spectral maximum of a signal against Gaussian white noise is considered. The sensitivity of the monitoring method based on measuring the frequency shifts of the field maximum is estimated. For a specific case of a Gaussian signal's spectrum and the medium perturbed by background internal waves, results of calculations are presented.

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INTRODUCTION

The interference fields of oceanic waveguides contain points at which the field intensity reaches local maxima due to the constructive interference of modes [1]. The presence of such points sensitive to perturbations of propagation conditions is a characteristic feature of wave fields in waveguide systems. Determination of the main laws that govern the space–frequency or time–frequency distributions of local maxima and investigation of their dynamics caused by changes in the propagation conditions represent an interesting problem, which offers qualitatively new possibilities for a wide range of applications. One such promising area of investigations of the fine space-frequency's interference pattern includes the studies of the interference invariant [2–5], which describes the frequency shifts of local field maxima as functions of the horizontal source-receiver's distance. New diagnostic possibilities are also offered by the method based on measuring the variations that occur in the frequency shifts of local interference maxima because of oceanic inhomogeneities [6, 7]. The method is founded on the mode dispersion, which leads to frequency shifts of the interference pattern due to the variability of the medium. The fruitfulness of this approach for solving the inverse problem (the approach was called sweep-monitoring) has been confirmed by the data of full-scale experiments [8] and computer simulations [9–13].

Considerable attention has been given to systems using the concept of phase conjugation [14] for compensating the effect of the inhomogeneities of the medium, as well as the effect of the field localization (or focusing) in multimode systems. In this case, the focusing is controlled by varying the reference frequency of transmission without changing the inverted field's distribution formed at the aperture in the absence of perturbation. The efficiency of the field's focusing control is discussed in [15–18], the possibilities of sweep-monitoring with the use focusing are analyzed in [10, 11, and 13], and the application of focusing for the reverberation signal's control is studied in [19–21].

Thus, determination of the varying frequency positions of local field maxima is an important problem for many areas of investigation in ocean acoustics. The problem of the minimal frequency shift of the interference pattern that allows resolution of neighboring local maxima is considered in deterministic form in [22]. At the same time, the inevitable presence of noise imposes basic limitations on the accuracy of the field maximum's indication and, hence, on the accuracy of measuring the frequency shifts of the field maxima. Therefore, investigation of the stability of the field maximum's frequency shifts with respect to the noise level is a topical problem. Unlike the classical problem of estimating the position of a pulse on the time axis [23], the present paper estimates the frequency position of the spectral maximum of a signal on the background of noise, which, evidently, requires special consideration.

The present paper discusses the error in determining the position of the spectral maximum and the frequency shift of this maximum because of the perturbation of the medium. This allows estimating the minimal fluctuation level of the oceanic inhomogeneity's parameters, which can be determined from the data on the frequency shifts of the local field's maxima. The results of this study are illustrated by an example of a Gaussian signal's spectrum in application to the model of perturbation by background internal waves.

STATEMENT OF THE PROBLEM

At the input of the receiving system, the spectral realization $\xi(\omega)$ is set in the form of the sum of the signal and noise spectra:

$$\xi(\omega) = s(\omega, \omega_0) + n(\omega). \quad (1)$$

Here, $s(\omega, \omega_0) = s(\omega - \omega_0)$ is the frequency spectrum of the signal with its maximum at the frequency $\omega = \omega_0$, where $\omega = 2\pi f$ is the cyclic frequency; $n(\omega)$ is the spectral amplitude of steady-state white noise: $\langle n(\omega)n^*(\omega')\rangle = N\delta(\omega - \omega')$, where N is the spectral density; the angular brackets denote statistical averaging over an ensemble of random realizations. To simplify the calculations, the spectrum of the signal is assumed to be symmetric about its maximum: $s(\omega - \omega_0) = s(\omega_0 - \omega)$. The frequency ω_0 can be estimated as in the case of determining the position of a pulse on the time axis [23], namely, with the use of matched processing. The difference is in that the realization of the cross-correlation function is between the chosen spectrum $\xi(\omega)$ given by Eq. (1) and the spectrum of the useful signal $s(\omega, \omega_*) = s(\omega - \omega_*)$:

$$J(\omega_*) = \int \xi(\omega)s^*(\omega, \omega_*)d\omega = g_s(\omega_*) + g_n(\omega_*), \quad (2)$$

where

$$g_s(\omega_*) = \int s(\omega, \omega_0)s^*(\omega, \omega_*)d\omega,$$

$$g_n(\omega_*) = \int n(\omega)s^*(\omega, \omega_*)d\omega.$$

Here, ω_* is the varied frequency corresponding to the maximum of the spectrum $s(\omega, \omega_*)$. The function $g_s(\omega_*)$ obtained at the output of the receiver is an autocorrelation function of the input's useful signal spectrum, and it can be called the signal's spectral function. The function $g_n(\omega)$ caused by noise is the cross-correlation function between the noise spectrum and the input spectrum of the useful signal, let us call it the noise's spectral function. The specific form of the function $g_n(\omega)$ due to noise $n(\omega)$ is different for different realizations (1). If we use the Fourier transforms

$$u(t) = \int u(\omega)\exp(i\omega t)d\omega,$$

$$u(\omega) = (1/2\pi)\int u(t)\exp(-i\omega t)dt,$$

the ratio of the maximal value of the signal's spectral function to the rms value of the noise's spectral function, i.e., the signal-to-noise ratio, will be

$$q = g_{s\max}(\omega_0)/\sigma_n = \sqrt{E/2\pi N}. \quad (3)$$

Here, $E = \int s^2(t)dt = 2\pi\int |s(\omega, \omega_0)|^2 d\omega$ is the signal energy. By analogy with correlation processing in the time domain, the device implementing algorithm (2) can be called the correlation (matched) receiver in the frequency domain.

If noise is absent, i.e., $n(\omega) = 0$, the maximum of the function $J(\omega_*)$ given by Eq. (2) corresponds to the maximum of the spectral function: $\omega_* = \omega_0$. In the presence of noise, the maximum of $J(\omega_*)$ occurs at $\omega_* = \hat{\omega}_0$ different from ω_0: $\max J(\omega_*) = J(\hat{\omega}_0)$. Perturbation of the medium causes a frequency shift $\Delta\omega$ of the signal spectrum's maximum. As a result, when noise is absent, the position of the signal spectrum's maximum will be at the frequency $\omega_1 = \omega_0 + \Delta\omega$. In the presence of noise, the maximum occurs at the frequency $\hat{\omega}_1$, so that the frequency shift of the maximum is $\Delta\hat{\omega} = \hat{\omega}_1 - \hat{\omega}_0$. The problem consists in determining the statistical fluctuations of the frequency $\tilde{\omega}_0 = \hat{\omega}_0 - \omega_0$ and estimating the sensitivity of the monitoring method based on measuring the frequency shift $\Delta\hat{\omega}$ of the local field's maximum.

SOLUTION OF THE PROBLEM

The most plausible estimate of ω_0 will be such a value of $\omega_* = \hat{\omega}_0$ that corresponds to the maximum of the cross-correlation function $J(\omega_*)$. It should satisfy the equation $\partial J(\omega_*)/\partial\omega_*|_{\omega_* = \hat{\omega}_0} = 0$. For brevity, in the subsequent calculations, the partial derivative is denoted as $\partial s(\omega, \omega_*)/\partial\omega_*|_{\omega_* = \hat{\omega}_0} = \partial s(\omega, \hat{\omega}_0)/\partial\omega_*$. Using Eq. (2), we obtain

$$\frac{\partial J(\hat{\omega}_0)}{\partial\omega_*} = \int [s(\omega, \omega_0) + n(\omega)]\frac{\partial s^*(\omega, \hat{\omega}_0)}{\partial\omega_*}d\omega = 0. \quad (4)$$

Under the low-noise assumption, we expand the function $\partial s^*(\omega, \hat{\omega}_0)/\partial\omega_*$ in a Taylor series in powers of $\tilde{\omega}_0 = \hat{\omega}_0 - \omega_0$:

$$\frac{\partial s^*(\omega, \hat{\omega}_0)}{\partial\omega_*} = \frac{\partial s^*(\omega, \omega_0)}{\partial\omega_*} + \frac{\partial^2 s^*(\omega, \omega_0)}{\partial\omega_*^2}\tilde{\omega}_0 + \frac{1}{2}\frac{\partial^3 s^*(\omega, \omega_0)}{\partial\omega_*^3}(\tilde{\omega}_0)^2 + \ldots. \quad (5)$$

In this series, we will only retain the linear term. Note that the derivative is $\partial s^*(\omega, \omega_0)/\partial\omega_* \neq 0$, whereas $\partial s^*(\omega, \omega_0)/\partial\omega = 0$. Substituting Eq. (5) into Eq. (4) and taking into account that $\int s(\omega, \omega_0)[\partial s^*(\omega, \omega_0)/\partial\omega_*]d\omega = 0$, we obtain the following estimate for the variance of frequency fluctuations of the spectral maximum's position $\tilde{\omega}_0$ in the first approximation:

$$\sigma^2_{\tilde{\omega}_0} = \langle(\hat{\omega}_0 - \omega_0)^2\rangle$$

$$= \frac{N\int\left|\frac{\partial s(\omega, \omega_0)}{\partial \omega_*}\right|^2 d\omega}{\left|\int s(\omega, \omega_0)\frac{\partial^2 s^*(\omega, \omega_0)}{\partial \omega_*^2}d\omega\right|^2 + N\int\left|\frac{\partial^2 s(\omega, \omega_0)}{\partial \omega_*^2}\right|^2 d\omega}. \quad (6)$$

This is a quantitative characteristic of the error in measuring the frequency ω_0. Here, the mathematical expectation is $m_{\tilde{\omega}_0} = \langle\hat{\omega}_0 - \omega_0\rangle = 0$. If the square of the signal-to-noise ratio (3) satisfies the condition $q^2 \gg 1$, by virtue of the Schwartz–Bunyakowsky inequality, Eq. (6) takes the form

$$\sigma^2_{\tilde{\omega}_0} = \frac{N\int\left|\frac{\partial s(\omega, \omega_0)}{\partial \omega_*}\right|^2 d\omega}{\left|\int s(\omega, \omega_0)\frac{\partial^2 s^*(\omega, \omega_0)}{\partial \omega_*^2}d\omega\right|^2}. \quad (7)$$

The second derivative in the integrand in Eq. (4) characterizes the steepness of the spectrum at the point $\omega = \omega_0$. Hence, in indicating the position of the spectral maximum, matched filter (2) realizes the maximal possible ratio of the spectrum steepness to the noise's spectral density. In estimating the acceptability of inequalities of the $a \gg b$ type, we proceed from the criterion that the left-hand and right-hand parts differ by no more than an order of magnitude, i.e., by a factor of 10. In view of notation for the signal-to-noise ratio (3), Eq. (7) can be represented in the form

$$\sigma^2_{\tilde{\omega}_0} = \frac{1}{q^2}\frac{\int\left|\frac{\partial s(\omega, \omega_0)}{\partial \omega_*}\right|^2 d\omega \int|s(\omega, \omega_0)|^2 d\omega}{\left|\int s(\omega, \omega_0)\frac{\partial^2 s^*(\omega, \omega_0)}{\partial \omega_*^2}d\omega\right|^2},$$

which is convenient to use for comparative analysis of the estimates obtained for the variances of different signal spectra in the case of identical signal-to-noise ratios. Result (6) can also be obtained by the small-parameter method based on the expansion of cross-correlation function (2) in inverse powers of the signal-to-noise ratio. This approach was used to determine the variance of the estimated temporal position of a pulse [24]. The present paper gives a briefer derivation, which allows a simple and illustrative formulation of the applicability condition for linear expansion (5).

For this purpose, we substitute expansion (5) in Eq. (4) by retaining the quadratic term and take into account the Gaussian's factorization property of the mean value of a product [25]. As a result, for estimating the variance $\sigma^2_{\tilde{\omega}_0}$ in the second-order approximation, we obtain the quadratic equation

$$a\sigma^4_{\tilde{\omega}_0} + 2b\sigma^2_{\tilde{\omega}_0} - 2c = 0, \quad (8)$$

where

$$a = N\int\left|\frac{\partial^3 s^*(\omega, \omega_0)}{\partial \omega_*^3}\right|^2 d\omega,$$

$$c = N\int\left|\frac{\partial s(\omega, \omega_0)}{\partial \omega_*}\right|^2 d\omega, \quad (9)$$

$$b = \left|\int s(\omega, \omega_0)\frac{\partial^2 s^*(\omega, \omega_0)}{\partial \omega_*^2}d\omega\right|^2 + N\int\left|\frac{\partial^2 s(\omega, \omega_0)}{\partial \omega_*^2}\right|^2 d\omega.$$

The second-order approximation for the variance, which is determined by the solution to Eq. (8), asymptotically tends to the variance estimate given by Eq. (6) when the condition

$$b^2 \gg 2ac \quad (10)$$

is satisfied. This condition can be considered as the criterion of applicability of the first approximation.

In the presence of high-level noise, the above approach allows one to obtain higher-order approximations for estimating the variance of the frequency position of the spectral maximum and, in every specific case, to determine the limits of applicability of the chosen approximation (which is very important). The drawback of the given method consists in that, as the number of terms in expansion (5) increases, calculation of the variance $\sigma^2_{\tilde{\omega}_0}$ becomes rather difficult. The approach is effective if the expansion is restricted to the fourth-order term (the fourth derivative), since the inclusion of higher-order approximations requires application of numerical methods for solving the algebraic equations.

In the general case, higher-order approximations presumably lead to ambiguity of the variance estimate $\sigma^2_{\tilde{\omega}_0}$; in other words, to ambiguity in indicating the position of the maximum of cross-correlation function (2). Physically, this means the presence of several peaks in this function, one of the peaks being true and the other peaks being false. The ambiguity in determining the position of the maximum is eliminated by increasing the signal-to-noise ratio. Therefore, in most of the physical applications, it is sufficient to consider the first approximation.

Assuming that, at a small frequency shift $\Delta\omega$ caused by an inhomogeneity, the form of the signal spectrum is retained (this assumption is quite permissible), we can estimate the mean square fluctuation of $\Delta\hat{\omega}$, i.e., $\langle(\Delta\tilde{\omega})^2\rangle = \langle(\Delta\hat{\omega} - \Delta\omega)^2\rangle$, as

$$\sigma^2_{\Delta\tilde{\omega}} = \sigma^2_{\tilde{\omega}_0} + \sigma^2_{\tilde{\omega}_1} \approx 2\sigma^2_{\tilde{\omega}_0}, \quad (11)$$

so that the variance of the frequency shift does not depend on its value. Here, $\sigma^2_{\tilde{\omega}_1}$ is the estimate of the fluctuation variance for the frequency $\tilde{\omega}_1$ corresponding to the spectral maximum.

The threshold sensitivity of sweep-monitoring is understood as such a frequency shift $\Delta\omega_{th}$ of the local maximum that is identical to the rms value (standard fluctuation) of $\Delta\hat{\omega}$, i.e., $\Delta\omega_{th} = \sigma_{\Delta\tilde{\omega}}$. Hence, according to Eq. (11), in the first approximation of dispersion estimate (6), the limiting resolution of neighboring maxima is

$$(\Delta\omega_{th})^2 = \frac{2N\int\left|\dfrac{\partial s(\omega, \omega_0)}{\partial\omega_*}\right|^2 d\omega}{\left|\int s(\omega, \omega_0)\dfrac{\partial^2 s^*(\omega, \omega_0)}{\partial\omega_*^2}d\omega\right|^2 + N\int\left|\dfrac{\partial^2 s(\omega, \omega_0)}{\partial\omega_*^2}\right|^2 d\omega}. \quad (12)$$

Thus, the measured frequency shift $\Delta\omega$ should be sufficiently large to provide the excess over the threshold: $\Delta\omega \geq \gamma\Delta\omega_{th}$, $\gamma \geq 1$. The value of the coefficient γ is chosen to provide the most accurate measurement of the field maximum's frequency shifts. The oceanic inhomogeneity's model being known, Eq. (12) allows one to estimate the error in the perturbation parameters determined from the data on the frequency shifts of interference maxima, i.e., from solving the inverse problem.

APPLICATION OF THE RESULTS

Let us consider a Gaussian spectrum

$$s(\omega, \omega_0) = A\exp\left[\frac{-(\omega - \omega_0)^2}{2p^2}\right], \quad (13)$$

for which it is possible to obtain the results in analytical form. Here, A is the spectrum amplitude and $p = \delta\nu/2\sqrt{2}$, where $\delta\nu$ is the spectrum width at a level of $1/e$ of the spectrum maximum. The latter quantity characterizes the steepness of the spectrum at the point $\omega = \omega_0$: $s(\omega_0, \omega_0)/[\partial^2 s(\omega_0, \omega_0)/\partial\omega^2] = (\delta\nu)^2/8$. By virtue of Eq. (3), the square of the signal-to-noise ratio is

$$q^2 = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{A^2\delta\nu}{N}.$$

Substituting Eq. (13) in Eq. (12), we estimate the threshold sensitivity of the method. As a result, we obtain

$$(\Delta\omega_{th})^2 = 2\left(\frac{\delta\nu}{2q}\right)^2\left(1 + \frac{3}{q^2}\right). \quad (14)$$

If $q^2 \gg 3$, i.e., $q \geq 5.5$, Eq. (14) takes the form

$$\Delta\omega_{th} = \frac{\delta\nu}{\sqrt{2}q}. \quad (15)$$

Thus, at a given signal-to-noise ratio, an increase in the spectrum width reduces the sensitivity of the method. According to Eqs. (9) and (10), estimate (14) of the threshold frequency's shift is valid under the condition that

$$q^4[1 + (3/q^2)]^2 \gg 30, \quad \text{i.e.,} \quad q \geq 3.8,$$

whereas estimate (15) of the threshold frequency's shift is valid when

$$q^4 \gg 30, \quad \text{i.e.,} \quad q \geq 4.2.$$

One can see that the first approximation of the threshold sensitivity is justified under the condition that the signal-to-noise ratio exceeds several units.

For illustration, let us estimate of the sensitivity of the method by considering the propagation conditions and the model reconstruction of the frequency spectrum of internal waves with the use of field focusing by the conjugate wave's front [10]. For modeling, we use the focusing frequency $f_0 = 230$ Hz and the width of the focal spot $\delta\nu/2\pi \approx 16.5$ Hz. Assuming that the signal-to-noise ratio is $q = 5$ (14.0 dB), for the threshold frequency's shift given by Eq. (14), we obtain $\Delta f_{th} \approx$ 2.4 Hz. The frequency shift Δf caused by the displacements of liquid layers $\Delta\zeta(z_0)$ at the depth z_0 is $\Delta f = \kappa^{-1}(f_0, z_0)\Delta\zeta(z_0)$. For the depth $z_0 = 60$ m, the coefficient is $\kappa \approx 0.053$ m/Hz. The sound velocity's increments $\Delta c(z_0)$ are related to the frequency shift Δf by the formula $\Delta c(z_0) = -\kappa(f_0, z_0)[d\bar{c}(z_0)/dz]\Delta f$, where $\bar{c}(z)$ is the unperturbed sound's velocity profile [22]. Hence, the threshold's oscillation amplitude is $\Delta\zeta_{th} \approx 0.13$ m and the threshold's sound velocity is $\Delta c_{th} \approx 0.027$ m/s. Evidently, an increase in the signal-to-noise ratio raises the sensitivity of the method.

CONCLUSIONS

For Gaussian white noise, the fluctuation variance of the frequency corresponding to the spectral maximum of a signal is estimated in the first approximation. The condition under which this approximation should be sufficient is discussed. The approach allowing calculation of the variance with the use of higher approximations is demonstrated. The threshold sensitivity of the monitoring method based on the data on frequency shifts of the local field's maximum is estimated. The results of the study are illustrated by considering the perturbation of the medium by background internal waves as an example.

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