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Generalization of Biot's Equations with Allowance for Shear Relaxation of a Fluid[1]

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Abstract—On the basis of the generalized variational principle for dissipative continuum mechanics, a system of generalized Biot's equations is derived to describe the wave propagation in a two-phase porous permeable medium in the presence of shear relaxation in the pore-filling fluid. It was shown that the inclusion of shear viscoelasticity of the fluid leads to the appearance of two transverse modes in addition to two longitudinal modes described by the Biot theory. One of the transverse modes is an acoustic mode, whereas the other is a diffusion mode characterized by the linear frequency dependence of phase velocity and attenuation coefficient in the low-frequency region.

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INTRODUCTION

Today, heavy oil is an object of special interest for the oil industry. Heavy oil deposits exhibit a rheological oil behavior, so that a correct description of the rheological properties of heavy oil in a porous medium is important for optimizing the methods of deposit exploitation. The propagation of waves in porous permeable media is described by the Biot theory [1, 2]. Its various aspects have been considered in numerous publications (see, e.g., [3–5] and the literature cited there). The Biot theory is based on the conventional variational principle for nondissipative mechanical systems. The fluid filling the pores of the skeleton is considered as an ideal fluid. This description does not take into account the rheological properties of actual heavy oil, and, therefore, attempts were made to modify the original Biot theory. For example, in the last few years, publications appeared wherein viscosity relaxation was taken into account by generalizing the Biot operator for the dissipative function [6, 7]. However, this approach is not completely adequate, because it ignores the additional degree of freedom related to shear elasticity of the viscous fluid. This degree of freedom could not be introduced in the conventional variational principle for nondissipative mechanics, which lies at the heart of the Biot theory [1, 2]. Still, the necessary generalizations can be obtained with the use of the generalized variational principle for dissipative mechanical systems.

In my previous publications [8–11], the generalized variational principle for dissipative continuum mechanics was formulated as a simple combination of the Hamilton and Onsager variational principles. The generalized principle allows the derivation of the system of equations for dissipative hydrodynamics. The generalized variational principle can be formulated in terms of the mechanical and temperature displacement fields on the basis of the Lagrangian given in the form [8–11]

$$L = K - F - \int_0^t D dt,$$

where K and F are the kinetic and free energies and D is the dissipative function.

In the present paper, the proposed generalized variational principle is used, firstly, for introduction of shear viscosity in the equations of motion of single-phase hydrodynamics and, secondly, to generalize the Biot equations for a two-phase porous permeable medium. Such a generalization is necessary for describing the behavior of heavy oil in a porous medium, becouse heavy oils demonstrate elastic properties at high frequencies.

1. THE INCLUSION OF SHEAR RELAXATION FOR A SINGLE-PHASE FLUID

In the previous publications [8–11], it was shown that, by introducing additional internal parameters in the generalized principle by analogy with the Mandelshtam–Leontovich approach [12], it is possible to introduce the bulk viscosity in the equation of motion of the fluid and even to describe its relaxation.

It is of interest to consider how the shear viscosity is introduced in the equation of motion. In this connection, it should be noted that the additional terms that appear in the quadratic forms for free energy and dissipative function and are related to a certain internal parameter ξ imply its that its nature is scalar nature. However, such an internal parameter can also possess vector and tensor properties. In the last case the additional terms, related to a tensor internal parameter ξ_{ik}, appear in the expression for the free energy and it has the form

$$2F(\nabla\mathbf{u}, \xi_{ik}) = 2\mu\varepsilon_{ik}^2 + \lambda\varepsilon_{ll}^2 + a_1\xi_{ll}^2 + a_2\xi_{ik}^2 + 2b_1\xi_{kk}\varepsilon_{ll} + 2b_2\xi_{ik}\varepsilon_{ki}. \tag{1.1}$$

The kinetic energy is given by the conventional expression

$$2K(\dot{\mathbf{u}}) = \rho_0\dot{\mathbf{u}}^2, \tag{1.2}$$

and the dissipative function in the absence of the temperature component can be represented as

$$2D(\dot{\xi}_{ij}) = \gamma_1\dot{\xi}_{ll}^2 + \gamma_2\dot{\xi}_{ik}^2. \tag{1.3}$$

In the above expressions, $\mathbf{u}$ is the field of the mean mass displacements of the medium, $\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$ is the strain tensor, ρ_0 is the density of the medium, and λ and μ are its Lame constants.

In this description, the viscous fluid is initially considered as an elastic medium. The temperature displacement field, which is part of the generalized variational principle [8–11], is omitted for simplicity.

The system of equations of motion is obtained on the basis of the generalized principle by varying the action with the Lagrangian given above. In the case under consideration, this system can be represented in the form

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\mathbf{u}}} - \nabla\frac{\partial F}{\partial \nabla\mathbf{u}} = -\frac{\partial D}{\partial \dot{\mathbf{u}}},$$

$$\frac{\partial D}{\partial \dot{\xi}_{ik}} - \frac{\partial F}{\partial \xi_{ik}} = 0.$$

In view of Eqs. (1.1)–(1.3), the explicit form of the equations is as follows:

$$\rho_0\frac{d}{dt}\dot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu)\mathrm{grad}\,\mathrm{div}(\mathbf{u}) - b_1\mathrm{grad}\xi_{ll} - b_2\frac{\partial\xi_{ik}}{\partial x_k} = 0, \tag{1.4}$$

$$\gamma_1\delta_{ik}\frac{d\xi_{ll}}{dt} + \gamma_2\frac{d\xi_{ik}}{dt} + a_1\delta_{ik}\xi_{ll} + a_2\xi_{ik} + b_1\delta_{ik}\mathrm{div}\mathbf{u} + b_2\varepsilon_{ik} = 0. \tag{1.5}$$

Here, the first equation is the equation of motion, in which, for simplicity, the tensor notation is left for the vector that is obtained by taking the divergence of the tensor internal parameter. The second equation is the kinetic equation for the tensor internal parameter ξ_{ik}.

Convolving the kinetic equation by indices, we obtain a separate kinetic equation for the spherical part of the internal parameter tensor ξ_{ll}:

$$\tilde{\gamma}\frac{d\xi_{ll}}{dt} + \tilde{a}\xi_{ll} + \tilde{b}\varepsilon_{ll} = 0, \tag{1.6}$$

where the coefficients marked with tilde have the form

$$\tilde{\gamma} = 3\gamma_1 + \gamma_2, \quad \tilde{a} = 3a_1 + a_2, \quad \tilde{b} = 3b_1 + b_2.$$

The solution to kinetic equation (1.6) is expressed by the formula

$$\xi_{ll} = -\frac{\tilde{b}}{\tilde{\gamma}}\int_{-\infty}^{t} e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\varepsilon_{ll}(t')dt'. \tag{1.7}$$

For the remaining components of the internal parameter tensor ξ_{ik}, we also obtain a kinetic equation of form (1.6), with the only difference being that the solution to kinetic equation (1.7) is added to the inhomogeneous terms:

$$\gamma_2\frac{d\xi_{ik}}{dt} + a_2\xi_{ik} + b_2\varepsilon_{ik} + \tilde{a}_1\delta_{ik}\xi_{ll} + \tilde{b}_1\delta_{ik}\varepsilon_{ll} = 0, \tag{1.8}$$

where

$$\tilde{a}_1 = \left(a_1 - \tilde{a}\frac{\gamma_1}{\tilde{\gamma}}\right), \quad \tilde{b}_1 = \left(b_1 - \tilde{b}\frac{\gamma_1}{\tilde{\gamma}}\right).$$

Again, the solution to Eq. (1.8) has the a form analogous to Eq. (1.7) with allowance for the additional contributions of the terms with the factors $\tilde{a}_1$ and $\tilde{b}_1$. This solution is as follows:

$$\xi_{ik} = -\frac{b_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')} \times \left(\varepsilon_{ik} - \delta_{ik}\varepsilon_{ll}\left(1 - \frac{\tilde{b}}{b_2}\frac{(a_1\gamma_2 - a_2\gamma_2)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\right) - \delta_{ik}\frac{\tilde{b}(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{\tilde{\gamma}(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\int_{-\infty}^{t} dt' e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\varepsilon_{ll}. \tag{1.9}$$

Taking the divergence of tensor (1.9), we obtain the vector

$$\frac{\partial\xi_{ik}}{\partial x_k} = -\frac{b_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')} \times \left(\frac{1}{2}(\Delta\mathbf{u} + \nabla(\nabla\mathbf{u})) - \nabla(\nabla\mathbf{u})\left(1 - \frac{\tilde{b}}{b_2}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\right) - \frac{\tilde{b}(a_1\tilde{\gamma} - \tilde{a}\gamma_1)}{\tilde{\gamma}(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\int_{-\infty}^{t} dt' e^{-\frac{\tilde{a}}{\tilde{\gamma}}(t-t')}\nabla(\nabla\mathbf{u}). \tag{1.10}$$

Substituting Eqs. (1.10) and (1.7) in the first of Eqs. (1.4), we represent the latter as

$$\rho_0\frac{d}{dt}\dot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\nabla\mathbf{u})$$

$$= -\frac{\tilde{b}}{\tilde{\tilde{\gamma}}}\left(b_1 - b_2\frac{(a_1\tilde{\gamma}-\tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\int_{-\infty}^{t} dt' e^{-\frac{\tilde{a}}{\tilde{\tilde{\gamma}}}(t-t')}\nabla(\nabla\mathbf{u})$$

$$-\frac{b_2^2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')}$$

$$\times\left(\frac{1}{2}(\Delta\mathbf{u}+\nabla(\nabla\mathbf{u})) - \nabla(\nabla\mathbf{u})\left(1 - \frac{\tilde{b}}{b_2}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right)\right).$$

In the low-frequency limit, at a time exceeding the relaxation time, i.e., $t \gg \tilde{\gamma}/\tilde{a}$ and $t \gg \gamma_2/a_2$, the main contribution to the integrals is made by the vicinity of the upper limit $t' \approx t$. Therefore, applying the corresponding expansions, we obtain an equation analogous to the Navier–Stokes equation with the shear and bulk viscosities:

$$\rho_0\frac{d}{dt}\dot{\mathbf{u}} - \tilde{\mu}\Delta\mathbf{u} - (\tilde{\lambda}+\tilde{\mu})\operatorname{grad}\operatorname{div}(\mathbf{u}) = \tilde{\eta}\Delta\dot{\mathbf{u}} + \tilde{\zeta}\operatorname{grad}\operatorname{div}\dot{\mathbf{u}}, \tag{1.11}$$

where the effective elastic moduli and the shear and bulk viscosity coefficients are expressed by the formulas

$$\tilde{\mu} = \mu - \frac{b_2^2}{2a_2},$$

$$\tilde{\lambda} = \lambda + \frac{b_2^2}{2a_2} - \frac{\tilde{b}}{\tilde{\tilde{a}}}\left(b_1 - b_2\frac{(a_1\tilde{\gamma}-\tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right), \tag{1.12}$$

$$\tilde{\zeta} + \frac{\tilde{\eta}}{3} = \tilde{\gamma}\frac{\tilde{b}}{\tilde{a}^2}\left(b_1 - b_2\frac{(a_1\tilde{\gamma}-\tilde{a}\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right) - \gamma_2\frac{b_2}{a_2^2}\left(\frac{b_2}{2} - \tilde{b}\frac{(a_1\gamma_2 - a_2\gamma_1)}{(\tilde{a}\gamma_2 - a_2\tilde{\gamma})}\right), \tag{1.13}$$

$$\tilde{\eta} = \frac{1}{2}\gamma_2\frac{b_2^2}{a_2^2}.$$

It is important to note that the structure of the effective shear modulus $\tilde{\mu}$ involved in Eq. (1.12) is determined by the difference that can be zero. In the latter case, Eq. (1.11) becomes fully equivalent to the linearized Navier–Stokes equation for viscous fluids. For $\tilde{\mu} > 0$, we obtain the case of an elastic body with shear viscosity (the Voigt model) or with relaxation in a more general case. Thus, in terms of a unite approach, it is possible to describe viscous fluids and solid bodies with viscoelastic properties.

2. THE BIOT EQUATIONS ON THE BASIS OF THE GENERALIZED VARIATIONAL PRINCIPLE

Now, let us consider the derivation of the system of motion equation for a two-phase porous permeable medium on the basis of the generalized variational principle. The temperature field component is ignored by the Biot approach, and, therefore, the system considered below is assumed to be at a constant temperature. In terms of the Biot approach, a porous permeable medium is represented by two mutually penetrating continua describing the displacement fields of the porous elastic skeleton $\mathbf{u}_1$ and the fluid $\mathbf{u}_2$. According to Biot [1], the kinetic energy of such a system is a positive definite quadratic form of velocities of these fields:

$$2K(\dot{\mathbf{u}}) = \rho_{11}\dot{\mathbf{u}}_1^2 + 2\rho_{12}\dot{\mathbf{u}}_1\dot{\mathbf{u}}_2 + \rho_{22}\dot{\mathbf{u}}_2^2. \tag{2.1}$$

The free energy of the fluid without inclusion of internal parameters is also represented by positively determined quadratic form:

$$2F(\nabla\mathbf{u}_1, \nabla\mathbf{u}_2) = 2\mu_{11}\varepsilon_{ik}^2 + \lambda_{11}\varepsilon_{ll}^2 + \lambda_{22}(\nabla\mathbf{u}_2)^2 + 2\lambda_{12}\varepsilon_{ll}\nabla\mathbf{u}_2, \tag{2.2}$$

where ε_{ik} is the strain tensor of the elastic medium.

The dissipative function also is a positive definite quadratic form, which should be zero in the absence of relative motion of the fluid and the porous medium:

$$2D(\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2) = \beta(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2)^2. \tag{2.3}$$

The equations of motion obtained on the basis of the variational principle in the presence of two displacement fields, $\mathbf{u}_1$ and $\mathbf{u}_2$, are represented as

$$\frac{d}{dt}\frac{\partial K}{\partial\dot{\mathbf{u}}_1} - \nabla\frac{\partial F}{\partial\nabla\mathbf{u}_1} = -\frac{\partial D}{\partial\dot{\mathbf{u}}_1},$$

$$\frac{d}{dt}\frac{\partial K}{\partial\dot{\mathbf{u}}_2} - \nabla\frac{\partial F}{\partial\nabla\mathbf{u}_2} = -\frac{\partial D}{\partial\dot{\mathbf{u}}_2}.$$

With allowance for potentials (2.1)–(2.3), they take the form

$$\rho_{11}\frac{d}{dt}\dot{\mathbf{u}}_1 + \rho_{12}\frac{d}{dt}\dot{\mathbf{u}}_2 - \mu_{11}\Delta\mathbf{u}_1 - (\lambda_{11}+\mu_{11})\operatorname{grad}\operatorname{div}\mathbf{u}_1 - \lambda_{12}\operatorname{grad}\operatorname{div}\mathbf{u}_2 = -\beta(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2), \tag{2.4}$$

$$\rho_{22}\frac{d}{dt}\dot{\mathbf{u}}_2 + \rho_{12}\frac{d}{dt}\dot{\mathbf{u}}_1 - \lambda_{22}\operatorname{grad}\operatorname{div}\mathbf{u}_2 - \lambda_{12}\operatorname{grad}\operatorname{div}\mathbf{u}_1 = -\beta(\dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1). \tag{2.5}$$

One can see that Eqs. (2.4) and (2.5) are a complete analog of the well-known Biot equations. This allows us to determine all the coefficients appearing in the quadratic forms by the comparison with these equations. Indeed, Eqs. (2.4) and (2.5) can be represented in thea form that is conventional for the Biot theory:

$$\begin{aligned}&\rho_{11}\ddot{\mathbf{u}}_1+\rho_{12}\ddot{\mathbf{u}}_2+b(\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2)\\&=P\mathrm{grad}\,\mathrm{div}(\mathbf{u}_1)+Q\mathrm{grad}\,\mathrm{div}(\mathbf{u}_2)-\mu_{11}\mathrm{curl}\,\mathrm{curl}(\mathbf{u}_1),\end{aligned}\tag{2.4b}$$

$$\begin{aligned}&\rho_{22}\ddot{\mathbf{u}}_2+\rho_{12}\ddot{\mathbf{u}}_1+b(\dot{\mathbf{u}}_2-\dot{\mathbf{u}}_1)\\&=R\mathrm{grad}\,\mathrm{div}(\mathbf{u}_2)+Q\mathrm{grad}\,\mathrm{div}(\mathbf{u}_1),\end{aligned}\tag{2.5b}$$

where

$$P=\lambda_{11}+2\mu_{11},\quad Q=\lambda_{12},\quad R=\lambda_{22},\quad b=\beta.\tag{2.6}$$

Therefore, relations known from the Biot theory [5] can be used for the aforementioned quantities:

$$\begin{aligned}\rho_{11}&=(1-m)\rho_s+(a-1)m\rho_f,\\ \rho_{12}&=\rho_{21}=-(a-1)m\rho_f,\\ \rho_{22}&=am\rho_f,\\ b&=-m^2\eta/k,\\ P&=\lambda+2\mu-2\beta Mm+m^2M,\\ Q&=Mm(\beta-m),\\ R&=m^2M,\end{aligned}\tag{2.7}$$

where ρ_f and ρ_s are the densities of the fluid and the elastic skeleton, m is the porosity of the skeleton, η is the viscosity of the fluid, k is the permeability coefficient, λ and μ are the Lame constants of the elastic skeleton, $M=\rho_f c_f^2$ is the elastic modulus of the fluid, and c_f is its sound velocity. The mean density of the medium is given by the expression

$$\rho=(1-m)\rho_s+m\rho_f,$$

and, for the dimensionless coefficient responsible for the attached mass effect, the approximate formula can be used [5]:

$$a=\frac{1}{2}\left(\frac{1}{m}+1\right).$$

According to [1], the system of equations (2.4), (2.5) allows the existence of three independent modes of motion: two longitudinal waves and one transverse wave. Specifically, the transverse wave and one of the longitudinal waves are acoustic modes in the low-frequency limit, whereas the other longitudinal wave is a diffusion mode in the low-frequency limit.

3. GENERALIZATION OF THE BIOT EQUATIONS WITH ALLOWANCE FOR SHEAR RELAXATION OF THE FLUID

Generalization of the Biot equations with allowance for the relaxation characteristics of the fluid filling the pores is possible with the use of the approach described in the first section of this paper. Again, for simplicity and for direct comparison with the Biot theory, let us consider the medium at a constant temperature. When fluid relaxation is taken into account, the kinetic energy retains its form (2.1), whereas additional terms appear in the expression for the free energy due to introduction of the internal parameter for the fluid. As it was mentioned above, when taking into account shear relaxation, the fluid should initially be considered as an elastic medium. In view of this remark, let us represent the free energy in the general quadratic form

$$\begin{aligned}&2F(\nabla\mathbf{u}_1,\nabla\mathbf{u}_2,\xi_{ik})\\&=2\mu_{11}(\overset{1}{\varepsilon}_{ik})^2+\lambda_{11}(\overset{1}{\varepsilon}_{ll})^2+2\mu_{22}(\overset{2}{\varepsilon}_{ik})^2+\lambda_{22}(\overset{2}{\varepsilon}_{ll})^2\\&+2\mu_{12}\overset{1}{\varepsilon}_{ik}\overset{2}{\varepsilon}_{ik}+2\lambda_{12}\overset{1}{\varepsilon}_{ll}\overset{2}{\varepsilon}_{kk}\\&+a_1\xi_{ll}^2+a_2\xi_{ik}^2+2b_1\xi_{kk}\overset{2}{\varepsilon}_{ll}\\&+2b_2\xi_{ik}\overset{2}{\varepsilon}_{ki}+2c_1\xi_{kk}\overset{1}{\varepsilon}_{ll}+2c_2\xi_{ik}\overset{1}{\varepsilon}_{ki}.\end{aligned}\tag{3.1}$$

Here, $\overset{1}{\varepsilon}_{ik}$ and $\overset{2}{\varepsilon}_{ik}$ are the strain tensors of the elastic skeleton and the fluid and ξ_{ik} is the tensor internal parameter. The dissipative function includes the terms due to both dissipation inside the viscous fluid and relative motion of the phases. It should be zero in the state of thermodynamic equilibrium, when internal macroscopic motions and relaxation processes are absent:

$$2D(\dot{\mathbf{u}}_1,\dot{\mathbf{u}}_2,\dot{\xi}_{ik})=\beta(\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2)^2+\gamma_1\dot{\xi}_{ll}^2+\gamma_2\dot{\xi}_{ik}^2.\tag{3.2}$$

It can be seen that, when the coefficients λ_{12}, μ_{12}, β, and ρ_{12} responsible for the interaction between the phases are zero, the reduced potentials separately describe the elastic medium and the viscous fluid with relaxation.

It is important to take into account shear relaxation in the interaction of the phases. Therefore, below, to simplify the formulas, we retain the terms containing the parameter ξ_{ik} and not its convolution ξ_{ll} by setting $a_1=0$, $b_1=0$, $c_1=0$, and $\gamma_1=0$. For such a system, the equations of motion have the form

$$\begin{aligned}\frac{d}{dt}\frac{\partial K}{\partial\dot{\mathbf{u}}_1}-\nabla\frac{\partial F}{\partial\nabla\mathbf{u}_1}&=-\frac{\partial D}{\partial\dot{\mathbf{u}}_1},\\ \frac{d}{dt}\frac{\partial K}{\partial\dot{\mathbf{u}}_2}-\nabla\frac{\partial F}{\partial\nabla\mathbf{u}_2}&=-\frac{\partial D}{\partial\dot{\mathbf{u}}_2},\\ \frac{\partial D}{\partial\dot{\xi}_{ik}}-\frac{\partial F}{\partial\xi_{ik}}&=0.\end{aligned}$$

Substituting potentials (2.1), (3.1), and (3.2) in these equations, we obtain

$$\rho_{11}\ddot{\mathbf{u}}_1 + \rho_{12}\ddot{\mathbf{u}}_2 - \mu_{11}\Delta\mathbf{u}_1 - \mu_{12}\Delta\mathbf{u}_2$$
$$-(\lambda_{11}+\mu_{11})\nabla(\nabla\mathbf{u}_1) - (\lambda_{12}+\mu_{12})\nabla(\nabla\mathbf{u}_2) \quad (3.3)$$
$$= -\beta(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) + c_2\frac{\partial\xi_{ik}}{\partial x_k},$$

$$\rho_{22}\ddot{\mathbf{u}}_2 + \rho_{12}\ddot{\mathbf{u}}_1 - \mu_{22}\Delta\mathbf{u}_2 - \mu_{12}\Delta\mathbf{u}_1$$
$$-(\lambda_{22}+\mu_{22})\nabla(\nabla\mathbf{u}_2) - (\lambda_{12}+\mu_{12})\nabla(\nabla\mathbf{u}_1) \quad (3.4)$$
$$= -\beta(\dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1) + b_2\frac{\partial\xi_{ik}}{\partial x_k},$$

$$\gamma_2\dot{\xi}_{ik} + a_2\xi_{ik} + b_2\varepsilon^2_{ik} + c_2\varepsilon^1_{ik} = 0. \quad (3.5)$$

As it was shown earlier, in the absence of elastic relaxation in the fluid $b_2 = 0$ and in the elastic skeleton $c_2 = 0$, Eqs. (3.3) and (3.4) represent an analog of the well-known Biot equations for two elastic continua. If, in addition, we set $\mu_{22} = 0$ and $\mu_{12} = 0$, we obtain the system of Biot equations for a porous medium filled with a nonviscous fluid. As it was shown above, this allows us to determine all the remaining coefficients appearing in the quadratic forms by the direct comparison of the equations.

Equation (3.5) is analogous to Eq. (1.6), and its solution has the form

$$\xi_{ik} = -\frac{1}{\gamma_2}\int_{-\infty}^{t} e^{-\frac{a_2}{\gamma_2}(t-t')}(b_2\varepsilon^2_{ik}(t') + c_2\varepsilon^1_{ik}(t'))dt'.$$

Substituting this solution on the right-hand sides of Eqs. (3.3) and (3.4), we represent the latter equations as

$$\rho_{11}\ddot{\mathbf{u}}_1 + \rho_{12}\ddot{\mathbf{u}}_2 - \mu_{11}\Delta\mathbf{u}_1 - \mu_{12}\Delta\mathbf{u}_2$$
$$-(\lambda_{11}+\mu_{11})\nabla(\nabla\mathbf{u}_1) - (\lambda_{12}+\mu_{12})\nabla(\nabla\mathbf{u}_2)$$
$$= -\beta(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) + \frac{1}{2}\frac{c_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')} \quad (3.3b)$$
$$\times (u_2(\Delta\mathbf{u}_2 + \nabla(\nabla\mathbf{u}_2)) + c_2(\Delta\mathbf{u}_1 + \nabla(\nabla\mathbf{u}_1))),$$

$$\rho_{22}\ddot{\mathbf{u}}_2 + \rho_{12}\ddot{\mathbf{u}}_1 - \mu_{22}\Delta\mathbf{u}_2 - \mu_{12}\Delta\mathbf{u}_1$$
$$-(\lambda_{22}+\mu_{22})\nabla(\nabla\mathbf{u}_2) - (\lambda_{12}+\mu_{12})\nabla(\nabla\mathbf{u}_1)$$
$$= -\beta(\dot{\mathbf{u}}_2 - \dot{\mathbf{u}}_1) + \frac{1}{2}\frac{b_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')} \quad (3.4b)$$
$$\times (b_2(\Delta\mathbf{u}_2 + \nabla(\nabla\mathbf{u}_2)) + c_2(\Delta\mathbf{u}_1 + \nabla(\nabla\mathbf{u}_1))).$$

Thus, the system of equations (3.3b), (3.4b) is a generalization of the system of Biot equations to the case wherein the fluid in the pores is a relaxing medium.

Let us consider the consequences of this generalization. It is possible to split the displacement fields $\mathbf{u}_1$ and $\mathbf{u}_2$ into longitudinal and transverse components:

$$\mathbf{u}_1 = \mathrm{grad}\varphi_1 + \mathrm{curl}\boldsymbol{\psi}_1, \quad \mathbf{u}_2 = \mathrm{grad}\varphi_2 + \mathrm{curl}\boldsymbol{\psi}_2.$$

Then, we sequentially apply the operators div and curl to Eqs. (3.3b) and (3.4b) and, changing to the frequency representation ($t \longrightarrow \omega$), we show that the scalar and vector displacement potentials satisfy to the following system of equations:

$$\left(\lambda_{11} + 2\mu_{11} - \frac{c_2^2}{i\omega\gamma_2 + a_2}\right)\Delta\varphi_1$$
$$+\left(\lambda_{12} + 2\mu_{12} - \frac{b_2c_2}{i\omega\gamma_2 + a_2}\right)\Delta\varphi_2 \quad (3.5a)$$
$$= -(\omega^2\rho_{11} + i\omega\beta)\varphi_1 - (\omega^2\rho_{12} + i\omega\beta)\varphi_2,$$

$$\left(\lambda_{12} + 2\mu_{12} - \frac{b_2c_2}{i\omega\gamma_2 + a_2}\right)\Delta\varphi_1$$
$$+\left(\lambda_{22} + 2\mu_{22} - \frac{b_2^2}{i\omega\gamma_2 + a_2}\right)\Delta\varphi_2 \quad (3.5b)$$
$$= -(\omega^2\rho_{12} - i\omega\beta)\varphi_1 - (\omega^2\rho_{22} + i\omega\beta)\varphi_2,$$

$$\left(\mu_{11} - \frac{1}{2}\frac{c_2^2}{i\omega\gamma_2 + a_2}\right)\mathrm{curl}\,\mathrm{curl}\boldsymbol{\psi}_1$$
$$+\left(\mu_{12} - \frac{1}{2}\frac{b_2c_2}{i\omega\gamma_2 + a_2}\right)\Delta\boldsymbol{\psi}_2 \quad (3.6a)$$
$$= (\omega^2\rho_{11} + i\omega\beta)\boldsymbol{\psi}_1 + (\omega^2\rho_{12} - i\omega\beta)\boldsymbol{\psi}_2,$$

$$\left(\mu_{12} - \frac{1}{2}\frac{b_2c_2}{i\omega\gamma_2 + a_2}\right)\mathrm{curl}\,\mathrm{curl}\boldsymbol{\psi}_1$$
$$+\left(\mu_{22} - \frac{1}{2}\frac{b_2^2}{i\omega\gamma_2 + a_2}\right)\mathrm{curl}\,\mathrm{curl}\boldsymbol{\psi}_2 \quad (3.6b)$$
$$= (\omega^2\rho_{12} - i\omega\beta)\boldsymbol{\psi}_1 + (\omega^2\rho_{22} + i\omega\beta)\boldsymbol{\psi}_2.$$

Thus, the complete system of equations is split onto two independent subsystems for the longitudinal and vector potentials, which describe the propagation of longitudinal and transverse waves, respectively.

If, in the field space, we introduce the vectors

$$\boldsymbol{\varphi} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \quad \boldsymbol{\psi} = \begin{pmatrix}\boldsymbol{\psi}_1\\ \boldsymbol{\psi}_2\end{pmatrix}, \quad (3.7)$$

the system of equations (3.5), (3.6) can be represented in the form of two independent matrix equations,

$$\hat{A}\Delta\boldsymbol{\varphi} + \hat{B}\boldsymbol{\varphi} = 0, \quad \hat{C}\,\mathrm{curlcurl}\boldsymbol{\psi} - \hat{B}\boldsymbol{\psi} = 0, \qquad (3.8)$$

where the matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ have the forms

$$\hat{A} = \begin{pmatrix} \lambda_{11} + 2\mu_{11} - \dfrac{c_2^2}{i\omega\gamma_2 + a_2} & \lambda_{12} + 2\mu_{12} - \dfrac{b_2c_2}{i\omega\gamma_2 + a_2} \\ \lambda_{12} + 2\mu_{12} - \dfrac{b_2c_2}{i\omega\gamma_2 + a_2} & \lambda_{22} + 2\mu_{22} - \dfrac{b_2^2}{i\omega\gamma_2 + a_2} \end{pmatrix},$$

$$\hat{B} = \begin{pmatrix} \omega^2\rho_{11} + i\omega\beta & \omega^2\rho_{12} - i\omega\beta \\ \omega^2\rho_{12} - i\omega\beta & \omega^2\rho_{22} + i\omega\beta \end{pmatrix},$$

$$\hat{C} = \begin{pmatrix} \mu_{11} - \dfrac{1}{2}\dfrac{c_2^2}{i\omega\gamma_2 + a_2} & \mu_{12} - \dfrac{1}{2}\dfrac{b_2c_2}{i\omega\gamma_2 + a_2} \\ \mu_{12} - \dfrac{1}{2}\dfrac{b_2c_2}{i\omega\gamma_2 + a_2} & \mu_{22} - \dfrac{1}{2}\dfrac{b_2^2}{i\omega\gamma_2 + a_2} \end{pmatrix}.$$

From the similar structure of Eqs. (3.8) or (3.6) at (3.7), one can see that, in the presence of viscoelasticity of the fluid (as in the Biot theory), two transverse modes can exist in addition to two longitudinal modes.

Indeed, each of the matrix equations (3.8) can be diagonalized by linear transformation of variables. By introducing a linear combination of the form $\vartheta = \varphi_1 + \varepsilon\varphi_2$, it is possible to separate the first equation of Eqs. (3.8) into two independent Helmholtz equations:

$$\Delta\vartheta_{1,2} + Kl_{1,2}^2(\omega)\vartheta_{1,2} = 0, \qquad (3.9)$$

where the squares of the wave numbers are given by the expressions

$$Kl_{1,2}^2(\omega) = \frac{(\omega^2\rho_{11} + i\omega\beta) + \alpha_{1,2}(\omega^2\rho_{12} - i\omega\beta)}{\tilde{\lambda}_{11} + \alpha_{1,2}\tilde{\lambda}_{12}} \qquad (3.10)$$

and the following notations are introduced for brevity:

$$\tilde{\lambda}_{11} = \lambda_{11} + 2\mu_{11} - \frac{c_2^2}{i\omega\gamma_2 + a_2},$$

$$\tilde{\lambda}_{12} = \lambda_{12} + 2\mu_{12} - \frac{b_2c_2}{i\omega\gamma_2 + a_2},$$

$$\tilde{\lambda}_{22} = \lambda_{22} + 2\mu_{22} - \frac{b_2^2}{i\omega\gamma_2 + a_2}.$$

The parameter $\alpha_{1,2}$ is the root of the quadratic equation

$$A\alpha^2 + B\alpha + C = 0, \qquad (3.11)$$

in which the coefficients are determined as

$$A = \tilde{\lambda}_{22} + \tilde{\lambda}_{12} + \frac{i\omega}{\beta}(\tilde{\lambda}_{22}\rho_{12} - \tilde{\lambda}_{12}\rho_{22}),$$

$$B = \tilde{\lambda}_{11} - \tilde{\lambda}_{22}$$

$$+ \frac{i\omega}{\beta}(\tilde{\lambda}_{22}\rho_{11} + \tilde{\lambda}_{12}\rho_{12} - \tilde{\lambda}_{11}\rho_{12} - \tilde{\lambda}_{12}\rho_{22}),$$

$$C = -(\tilde{\lambda}_{12} + \tilde{\lambda}_{11}) + \frac{i\omega}{\beta}(\tilde{\lambda}_{12}\rho_{11} - \tilde{\lambda}_{11}\rho_{12}).$$

In the low-frequency approximation, the first terms in expansions of the roots of Eq. (3.11) by frequency have the form

$$\alpha_1 = 1, \quad \alpha_2 = -\frac{\tilde{\lambda}_{11} + \tilde{\lambda}_{12}}{\tilde{\lambda}_{22} + \tilde{\lambda}_{12}}.$$

The corresponding wave numbers are expressed as

$$Kl_1^2(\omega) = \frac{\omega^2(\rho_{11} + \rho_{12})}{\tilde{\lambda}_{11} + \tilde{\lambda}_{12}}, \qquad (3.12)$$

$$Kl_2^2(\omega) = i\omega\beta\frac{\tilde{\lambda}_{11} + 2\tilde{\lambda}_{12} + \tilde{\lambda}_{22}}{\tilde{\lambda}_{11}\tilde{\lambda}_{22} - \tilde{\lambda}_{12}^2}. \qquad (3.13)$$

From these expressions, it follows that, in the low-frequency approximation, the first mode with wave number (3.12) is an acoustic mode, whereas the second mode is a diffusion one. A similar behavior of longitudinal modes is predicted by the Biot theory.

Now, let us determine the modes of transverse waves being the splitting result of the second of Eqs. (3.8). Introducing the linear combination $\boldsymbol{\varsigma} = \boldsymbol{\psi}_1 + \tau\boldsymbol{\psi}_2$, we separate Eq. (3.8) in two independent Helmholtz equations,

$$\mathrm{curlcurl}\boldsymbol{\varsigma}_{1,2} - Ks_{1,2}^2(\omega)\boldsymbol{\varsigma}_{1,2} = 0, \qquad (3.14)$$

where the squares of the wave numbers are given by the formula

$$Ks_{1,2}^2(\omega) = \frac{(\omega^2\rho_{11} + i\omega\beta) + \alpha_{1,2}(\omega^2\rho_{12} - i\omega\beta)}{\tilde{\mu}_{11} + \alpha_{1,2}\tilde{\mu}_{12}} \qquad (3.15)$$

and the parameter $\alpha_{1,2}$ is the root of quadratic equation (3.11) with the coefficients

$$A = \tilde{\mu}_{22} + \tilde{\mu}_{12} + \frac{i\omega}{\beta}(\tilde{\mu}_{22}\rho_{12} - \tilde{\mu}_{12}\rho_{22}),$$

$$B = \tilde{\mu}_{11} - \tilde{\mu}_{22} + \frac{i\omega}{\beta}(\tilde{\mu}_{22}\rho_{11} + \tilde{\mu}_{12}\rho_{12} - \tilde{\mu}_{11}\rho_{12} - \tilde{\mu}_{12}\rho_{22}),$$

$$C = -(\tilde{\mu}_{12} + \tilde{\mu}_{11}) + \frac{i\omega}{\beta}(\tilde{\mu}_{12}\rho_{11} - \tilde{\mu}_{11}\rho_{12}).$$

Here,

$$\tilde{\mu}_{11} = \mu_{11} - \frac{1}{2}\frac{c_2^2}{i\omega\gamma_2 + a_2},$$
$$\tilde{\mu}_{12} = \mu_{12} - \frac{1}{2}\frac{b_2c_2}{i\omega\gamma_2 + a_2}, \quad \tilde{\mu}_{22} = \mu_{22} - \frac{1}{2}\frac{b_2^2}{i\omega\gamma_2 + a_2}. \tag{3.16}$$

In the low-frequency approximation (as long as $\tilde{\mu}_{12}$ and $\tilde{\mu}_{22}$ remain finite at low frequencies), the first terms in the expansions of the roots of Eq. (3.11) by the frequency have the forms

$$\alpha_1 = 1, \quad \alpha_2 = -\frac{\tilde{\mu}_{11} + \tilde{\mu}_{12}}{\tilde{\mu}_{22} + \tilde{\mu}_{12}}.$$

The corresponding wave numbers are

$$Ks_1^2(\omega) = \frac{\omega^2(\rho_{11} + \rho_{12})}{\tilde{\mu}_{11} + \tilde{\mu}_{12}}, \tag{3.17}$$

$$Ks_2^2(\omega) = i\omega\beta\frac{\tilde{\mu}_{11} + 2\tilde{\mu}_{12} + \tilde{\mu}_{22}}{\tilde{\mu}_{11}\tilde{\mu}_{22} - \tilde{\mu}_{12}^2}. \tag{3.18}$$

Expressions (3.17) and (3.18) for the wave numbers of the transverse modes have the forms similar to that of the wave numbers of longitudinal waves (Eqs. (3.12) and (3.13)), and, in the general case, the transverse modes should be expected to have the properties similar to those of longitudinal modes. Specifically, at low frequencies, one of the longitudinal modes (mode (3.17)) should be an acoustic mode, whereas the other (mode (3.18)) should be a diffusion mode.

However, it should be remembered that, when describing the viscous fluid, we considered it as an elastic medium with relaxation. To obtain a viscous fluid from an elastic body, it is necessary to set

$$\mu_{22} = \frac{b_2^2}{2a_2}, \quad \mu_{12} = \frac{b_2c_2}{2a_2}. \tag{3.19}$$

In this case, the acoustic mode (3.17) remains acoustic while the behavior of the diffusion mode is qualitatively different. Indeed, in the low-frequency limit, under conditions (3.19), we have

$$\tilde{\mu}_{22} = \frac{1}{2}i\omega\gamma_2\left(\frac{b_2}{a_2}\right), \quad \tilde{\mu}_{12} = \frac{1}{2}i\omega\gamma_2\frac{b_2c_2}{a_2^2},$$

$$\tilde{\mu}_{11} = \mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2} + \frac{1}{2}i\omega\gamma_2\left(\frac{b_2}{a_2}\right)^2.$$

In this case, the roots of quadratic equation (3.11) have the following asymptotics:

$$\alpha_1 = 1, \quad \alpha_2 = -\frac{\left(\mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2}\right)}{i\omega\gamma_2/2}\frac{a_2^2}{b_2(b_2 + c_2)}.$$

Then, for the wave number given by Eq. (3.17), we obtain the expression

$$Ks_1^2(\omega) = \frac{\omega^2(\rho_{11} + \rho_{12})}{\mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2^2}}, \tag{3.17b}$$

and, instead of Eq. (3.18), we obtain

$$Ks_2^2(\omega) = \frac{\beta}{\gamma_2}\left(\frac{a_2}{b_2}\right)^2 + \frac{1}{2}\frac{i\omega}{\left(\mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2}\right)\frac{b_2}{b_2 + c_2}} \tag{3.18b}$$
$$\times\left(\beta + 2\frac{\rho_{12}}{\gamma_2}\left(\mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2}\right)\frac{a_2^2}{b_2(b_2 + c_2)}\right).$$

From Eq. (3.18b), it formally follows that the second transverse mode also appears to be a diffusion mode with a phase velocity dispersion linear in frequency:

$$c_{s2}(\omega) = \omega\sqrt{\frac{\gamma_2}{\beta}\frac{b_2}{a_2}},$$

which value depends on the ratio of the dissipative factors γ_2 and β. The attenuation coefficient also is linear in frequency:

$$\alpha_{s2}(\omega) = \frac{1}{4}\omega\sqrt{\frac{\beta}{\gamma_2}}\left(\frac{b_2}{a_2}\right)$$
$$\times\frac{\left(\beta + 2\frac{\rho_{12}}{\gamma_2}\left(\mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2}\right)\frac{a_2^2}{b_2(b_2 + c_2)}\right)}{\left(\mu_{11} - \frac{1}{2}\frac{c_2^2}{a_2}\right)\frac{b_2}{b_2 + c_2}}.$$

It should be noted that, if the dissipation coefficient β is set to be zero with the dissipative function being determined by the relaxation term alone, from Eq. (3.21) we obtain that the second shear mode again appears to be a diffusion mode with the square of the wave number

$$Ks_2^2(\omega) = i\omega\frac{\rho_{12}}{\gamma_2}\frac{a_2^2}{b_2^2},$$

which value is determined by ratio between the parameter ρ_{12} related to the attached mass and the dissipation factor γ_2.

CONCLUSIONS

It is shown in the paper, that natural introduction of shear viscosity in the hydrodynamic equations system can be achieved in the framework of the generalized variational principle by the introduction of the tensor internal parameter into the expressions for the free energy and dissipative function, in accordance with the Mandelshtam-Leontovich approach. In terms of other variational approaches, the introduction of shear viscosity usually causes difficulties. The proposed method allows generalization of the Navier–Stokes equation with allowance for viscosity relaxation and, hence, makes it possible to obtain a combined description of the inelastic behavior of viscous fluids and solids.

It should be noted that, since the bulk and shear viscosities are physical characteristics common to fluids and gases, the physical meaning of the internal parameter used for description of these properties should also be common for fluids and gases. Therefore, as the internal parameter, we can take only the most general structural characteristics of the medium, such as, e.g., the mean positions of atoms or molecules of the medium with respect to each other in the thermodynamic equilibrium state. The reducing of the mean distance between atoms and molecules should be related to volume relaxation and, hence, to bulk viscosity, while the reducing of spatial (angular) equilibrium at every point of the medium should be related to shear relaxation and, hence, shear viscosity. Thus, the tensor internal parameter introduced in this paper is of kinetic origin and can be associated with the order parameter, which is commonly used in the theory of phase transitions.

It was shown that the conventional system of Biot equations for a two-phase porous permeable medium can be immediately obtained on the basis of the generalized variational principle. However, if we take into account the additional degree of freedom related to the presence of transverse waves in the viscous fluid, the generalized variational principle will allow us to derive the system of equations of motion that generalize the system of Biot equations with allowance for this additional shear degree of freedom. It should be stressed that, in terms of the existing variational principle for nondissipative continuum mechanics, which underlies the Biot theory, the shear degree of freedom cannot be taken into account because of its fundamentally dissipative nature.

It was also shown that, with allowance for the shear viscoelasticity of the fluid, not only two longitudinal modes can exist, as in the Biot theory, but also two transversal modes can be exist in the medium as well. One of the transverse modes is an acoustic mode, whereas the other proves to be a diffusion mode with the phase velocity and the attenuation coefficient linearly depending on frequency in the low-frequency region.

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