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ACOUSTIC SIGNAL PROCESSING AND COMPUTER SIMULATION

On the Use of Discrete Wavelet Transform for Solving Integral Equations of Acoustic Scattering[1]

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Abstract—In this paper, the discrete wavelet transform (DWT) is used to solve the dense system of equations which arises from integral equation of acoustic scattering. The DWT using appropriate wavelet family for acquiring larger sparsification of the system matrix is used to obtain a sparse approximation to the transformed matrix that is used in place of the original matrix in an iterative solver. Alternatively DWT is also used to design sparse preconditioners for an iterative method. Also, DWT-based preconditioners are constructed to accelerate iterative Krylov subspace methods. Convergence rates and number of operations are discussed for each case.

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1. INTRODUCTION

Integral equation methods have been used to solve exterior acoustic radiation and scattering problems for many applications. In these problems, the external pressure is represented in terms of a distribution of an acoustic field on the surface of a scatterer or radiator. By forcing this representation to match a specified velocity distribution on the surface, an integral for the unknown source strengths is obtained. Once the source density is obtained, the pressure at any point in the exterior region can be computed. The surface Helmholtz integral equation is advantageous, for formulating the acoustic scattering problem, in that the problem's dimensionality is reduced by one and an infinite domain is transformed to finite boundaries in which the far-field radiation condition is satisfied.

As long as the 'length scale is comparable to the used wavelength, standard moment-method approaches are well suited for discretizing the integral equation of acoustic scattering problem [1]. The method of moment (MoM) is essentially a discretization scheme whereby a general operator equation is transformed into a matrix equation which can be solved numerically. This transformation is affected by projections on subspaces, which for acoustic scattering bodies are of finite dimensions. The resulting matrix is always dense when conventional expansion and testing functions are used. Recently, there has been much interest in using wavelets to sparsify that dense moment matrix [2, 3], and [4]. Extensive comparisons are conducted on different wavelet operators for various boundary integral equations in many works, as in [5] and [6].

This paper aims to efficiently solve the dense linear system arising from a Galerkin-type approximation of the boundary integral equation of acoustic scattering using Discrete Wavelet Transform (DWT). A sparse approximate linear system is obtained by DWT thresholding of the dense linear system. An iterative solver, such as the generalized minimum residual method (GMRES) [8], is then used to solve the sparsified linear system. The GMRES iterative method with restarts, GMRES(r) is known as an efficient method for solving non-Hermitian linear systems [9].

Although the smaller the threshold that is chosen the more accurate is the solution, but with an increased number of nonzero entries, thresholding introduces error. Alternatively, DWT based preconditioners for the dense linear system is used. Several DWT based preconditioners has been developed in the literature and has been found to be effective for a wide class of matrices [9–12]. These preconditioners, namely; the standard DWT, the DWT with permutation (DWTPer), and the Modified DWTPer (DWTPerMod), will be used for solving our dense linear system and their effects will be illustrated numerically.

For both the sparsified linear system and the DWT based peconditioned linear system, the number of iterations required to find a solution within a specified accuracy, the convergence rates of the residual and the time taken by the CPU will be considered as comparison minutes between different preconditioners.

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The comparisons are obtained for the problem of acoustic scattering on an acoustically hard sphere due to the availability of analytical solution. The availability of analytical solution helps in estimating the errors resulting from numerical and approximation techniques used.

2. METHOD OF MOMENT FORMULATION OF ACOUSTIC SCATTERING

The equivalent boundary integral formulation of acoustic scattering problem, which is valid for an acoustic medium B' exterior to a finite body B with surface S on which a unit normal $\mathbf{n}$, pointing into B', is defined. The body is submerged into an infinite linear acoustic medium. When a harmonic acoustic wave ϕ^i impinges upon the body B, the resulting integral equation for smooth boundaries has the following form;

$$C(P)\phi(P) = \int_S\left(\phi(Q)\frac{\partial\psi(P,Q)}{\partial n} - \psi(P,Q)\frac{\partial\phi(Q)}{\partial n}\right)dS_Q + 4\pi\phi^i(P). \quad (1)$$

This equation is called the surface Helmholtz integral equation where $\phi(P) = \phi(r_P)e^{i\omega t}$ at a point P and Q is a point on the body surface.

The free-space Green's function ψ for the Helmholtz wave Eq. (1) is given by

$$\psi(P,Q) = e^{-ikR}/R,$$

where R is the distance between the field point P and a source point Q, and n is the outward directed normal at Q. The coefficient $C(P)$ is defined at P on S provided that there is a unique tangent to S at such a P, as follows:

$$C(P) = \begin{cases} 0 \ \text{ for } \ P \in B' \\ 4\pi \ \text{ for } \ P \in B \\ 2\pi \ \text{ for } \ P \in S \end{cases}.$$

When P occupies a point on the surface S where there is no unique tangent plane ([13] and [14]).

Considering an axisymmetric body and applying hard-scattering boundary condition $\left(\frac{\partial\phi}{\partial n} = 0\right)$, the integral in Eq. (1) can be rewritten using a cylindrical coordinate system (ρ, θ, z) as follows:

$$\int_L \phi(Q)\left[\int_0^{2\pi}\frac{\partial}{\partial n}\left(\frac{e^{-ikR(P,Q)}}{R(P,Q)}\right)d\theta(Q)\right]\rho(Q)dL(Q), \quad (2)$$

where the axisymmetric assumption implies that the field $\phi(P)$ and its derivative are independent of $\theta(P)$ and the differential area element is defined as

$$dS(Q) = \rho(Q)d\theta(Q)dL(Q),$$

where $dL(Q)$ is the differential length of the generator L of the body at a surface point Q, where Q now is interpreted as an arbitrary point on L only.

The MoM method can be used, with orthogonal bases functions to approximate the unknown function ϕ, to convert the integral equation into a system of equations. For different node points ip and assuming the index of surface elements iq, the following discretized form of Eq. (1), for N nodes on the surface, can be written as follows

$$\mathbf{A}\phi = \mathbf{B}, \quad (3)$$

where $\mathbf{A}$ is an $N \times N$ matrix. ϕ and $\mathbf{B}$ are N vectors. An example for the hard scatterer where $\frac{\partial\phi}{\partial n} = 0$, we can write

$$\begin{aligned} A(i_p, i_q) &= \sum_{i_q=1}^{N} I_1\rho(i_q)dL(i_q), \ \ i_p \neq i_q, \\ A(i_p, i_p) &= \sum_{i_q=1}^{N} I_1\rho(i_q)dL(i_{1q}) - 2\pi, \ \ i_p = i_q, \end{aligned} \quad (4)$$

and

$$B(i_p) = -4\pi\phi^i(i_p) \ \ \forall i_p = 1\ldots N, \quad (5)$$

where ϕ is an N-dimension vector representing the field strength on the scatterer surface and ϕ^i is the incident field. Equation (3) is obtained using the collocation version of MoM which employs a delta function as a basis function.

3. DWT OF SYSTEM MATRIX

For electromagnetic problems, it was reported that almost identical results are obtained using Daubechies and wavelet-like bases [15] and [16]. Daubechies wavelets [17] are strictly localized in space and approximately localized in spatial frequency.

The wavelets can approximate finer resolutions near boundaries and corners of scattering surfaces. In general, classical wavelets seem to be good in computing low frequency scattering and antenna problems [18]. For these reasons and due to similar mathematical formulation of acoustic scattering, Daubechies wavelets are more appropriate for our problem. Many recent works employed Daubechies wavelets in solving scattering problems [19–22].

The standard DWT, based on Daubechies compactly supported orthogonal wavelets [20, 21] is a linear transformation that transforms a given smooth vector in the standard basis to a wavelet basis in which most coefficients may be small or nearly zero. Let $v = s^0 = (s_0^0, s_1^0, \ldots, s_{n-1}^0)^T$ be a vector of length n, such that $n = 2^k$ where k is an integer. Then the level $l \le k$ wavelet transform of v is defined by the following recurrence relations

$$s_j^{(l+1)} = \sum_{i=0}^{m-1} h_i s^l_{\langle i+2j\rangle_{n/2^l}}, \quad d_j^{(l+1)} = \sum_{i=0}^{m-1} g_i s^l_{\langle i+2j\rangle_{n/2^l}}, \quad (6)$$

where m is the order of compactly supported wavelets, $\langle m\rangle_n$ refers to m mod n, $h_0, h_1, \ldots, h_{m-1}$ are the m low-pass filter coefficients, and $g_0, g_1, \ldots, g_{m-1}$ are the high-pass filter coefficients derived from $h_0, h_1, \ldots, h_{m-1}$ as in [4] by the following relation

$$g_i = (-1)^i h_{m-1-i}, \quad (7)$$

$i = 0, 1, \ldots, m-1$. The $s_j^{(l+1)}$ represent weighted averages of the elements of $s^l_{\langle i+2j\rangle_{n/2^l}}$, $i = 0, 1, \ldots, m-1$ and the $d_j^{(l+1)}$ are weighted differences of the same elements. For a smooth vector, we expect the values of elements of $d_j^{(l+1)}$ to be small compared with that of $s_j^{(l+1)}$. In matrix form, a level k DWT of a vector v defined as follows

$$\tilde{v} = \mathbf{W}v, \quad (8)$$

where

$$\mathbf{W} = \mathbf{W_k W_{k-1} \ldots W_1} v, \quad (9)$$

and $\mathbf{W}_l$, $l = 1, 2, \ldots, k$ is an $n \times n$ orthogonal matrix. Implementation of the DWT and inverse DWT in terms of matrix multiplication would be expensive, so in practical the recurrence formulas (6) are used to implement the transformation and are referred to as Fast Wavelet Transform, FWT and inverse FWT. The cost of applying a level k transform FWT and Inverse FWT of an n-vector is $0(n\log_2 n)$. The 2-Dim FWT of an $n \times n$ matrix is made up of n vector FWTs in the column direction and n vector FWTs in the row direction. Hence the total cost of a level k 2-Dim FWT of an $n \times n$ matrix is $O(n^2\log_2 n)$.

The Galerkin's discretized form of the integral Eq. (3) is transformed using DWT, by the orthogonal wavelet transform matrix $\mathbf{W}$ as follows

$$\mathbf{WAW^TW}\phi = \mathbf{WB}, \quad (10)$$

where $\mathbf{A}$ is the Galerkin's moment matrix, ϕ is the unknown wavelet amplitudes vector, $\mathbf{B}$ is the incident field vector defined at the surface points. As such, Eq. (10) can be rewritten as

$$\tilde{\mathbf{A}}\tilde{\phi} = \tilde{\mathbf{B}}. \quad (11)$$

Since $\mathbf{W}$ is an orthogonal matrix, the spectrum and pseudo-spectra of $\tilde{\mathbf{A}}$ will be the same as those of $\mathbf{A}$, so convergence will not be affected by the transform.

A sparse approximation $\bar{\mathbf{A}}$ to the transformed matrix $\tilde{\mathbf{A}}$ in (11) is obtained by thresholding. That is, setting to zero all elements whose magnitude fall below a chosen threshold. The smaller the threshold the more accurate the approximation is, but at the cost of an increased number of nonzero entries. If A is not smooth enough, or there exist areas of non-smoothness, then the DWT will fail to give a sparse matrix. A possible measure of non-smoothness of a matrix $\mathbf{A}$ would be to consider finite differences along the rows and columns, see [10] and [12]. That is for each element $a_{i,j}$ of $\mathbf{A}$ $(i, j \ne n)$ we compute the infinity norm of the vector whose components are the differences $a_{i,j} - a_{i+1,j}$ and $a_{i,j} - a_{i,j+1}$. This norm is simple to implement and identifies areas of non-smoothness of $\mathbf{A}$.

If $\mathbf{A}$ is sufficiently smooth we are able to find a sparse approximation $\bar{\mathbf{A}}$ to $\tilde{\mathbf{A}}$, which can be used efficiently. Then, solving the sparse system

$$\bar{\mathbf{A}}\tilde{\phi} = \tilde{\mathbf{B}} \quad (12)$$

using the GMRES method without preconditioning. A solution $\tilde{\phi}$ can be, also, obtained directly by solving the original system of Eq. (3) after applying the inverse DWT.

However, approximately, $\bar{\mathbf{A}}$ is spectrally equivalent to the original matrix $\mathbf{A}$, that is, with GMRES the number of iterations are the same. Therefore, if the original matrix needs preconditioning, we need to precondition (12) a well.

4. WAVELET-BASED PRECONDITIONERS

In this section, we solve (11) using preconditioned GMRES method. The wavelet-based preconditioning looks for a sparse approximation $\mathbf{M}$ to $\tilde{\mathbf{A}}$, $\mathbf{M} \approx \tilde{\mathbf{A}}$ such that the following Eq. (13) has the same solution as (11), but with more favorable spectral properties.

$$\mathbf{M}^{-1}\tilde{\mathbf{A}}\tilde{\phi} = \mathbf{M}^{-1}\tilde{\mathbf{B}}. \quad (13)$$

The iterative linear' solver, GMRES method, preconditioned by $\mathbf{M}$, converges rapidly if $\mathbf{M}$ efficiently approximates $\tilde{\mathbf{A}}$ in some way, i.e. the matrices $\mathbf{M}^{-1}\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}\mathbf{M}^{-1}$ are close to the identity matrix. The wavelet-based preconditioners found in the literature namely; the standard DWT preconditioner with band cut [17], the DWT with permutation (DWTPer) preconditioner [10], and the modified DWT with permutation (DWTPerMod) preconditioner [23] are efficient preconditioners for the acoustic scattering problem, [10]. Mainly these types of preconditioners are

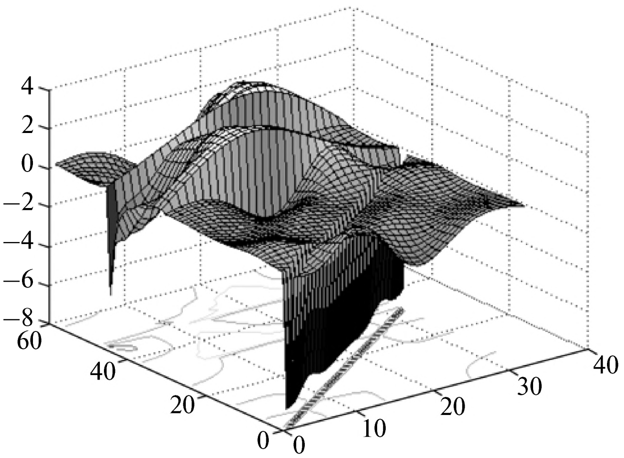

Fig. 1. Matrix **A** for $N = 32$.

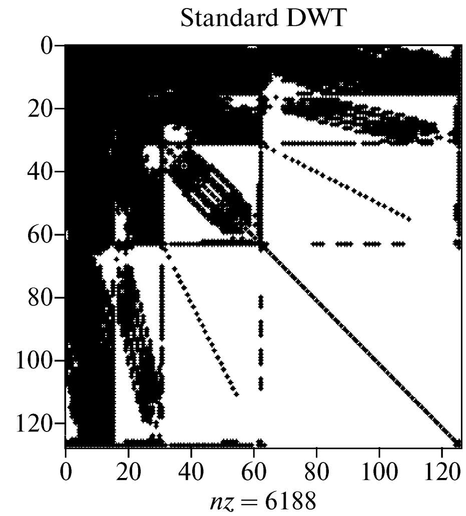

Fig. 2. Standard DWT with thresholding, $N = 128$.

based on the idea of splitting the matrix into the sum of a smooth matrix and a band matrix and then compressing the smooth part by means of a DWT. We review these preconditioners briefly.

Usually a 'finger'-like sparsity pattern results after a standard DWT is performed on a matrix **A** which is smooth but with non-smooth diagonal band. The pattern of thresholded transformed matrix, $\bar{\mathbf{A}}$, can be considered as a preconditioner which has the disadvantage of increasing number of nonzero entries during the LU factorization step of the GMRES iterative method. This is referred to as the fill-in property. To avoid this, a suitable band form **M** of $\bar{\mathbf{A}}$ is introduced [23]. Such a pattern may be used more advantageously than a finger-like one. Then, $\mathbf{M}^{-1}$ is used as the preconditioner to the linear system (11) solved using GMRES method. Thus, the solution to (11), ϕ, is found by applying the inverse DWT of $\tilde{\phi}$. This preconditioner is referred to as the standard DWT with band cut preconditioner.

Here the band size of **M** determines the cost of preconditioning step. For a small band size, the preconditioner may not approximate $\tilde{\mathbf{A}}^{-1}$ accurately, while increasing the size causes each iteration to be too expensive.

Alternatively in [10] a preconditioner was constructed by permuting the rows and columns of matrix $\tilde{\mathbf{A}}$, denoted by DWTPer, to avoid the creation of finger pattern matrices and form a banded matrix. The permuted DWT would give

$$\hat{\mathbf{A}} = \hat{\mathbf{W}}\mathbf{A}\hat{\mathbf{W}}^{\mathrm{T}}, \tag{14}$$

where

$$\hat{\mathbf{W}} = \mathbf{PW} \tag{15}$$

and **P** is an $n \times n$ permutation matrix. This implies that the permuted DWT can be implemented either directly using (15) or indirectly using P after a standard DWT has been applied. A band form **M** of $\hat{\mathbf{A}}$ is selected with an increase in bandwidth by at most $m(2^{L-1} - 1)$, for a level L DWTPer, as in [10]. Such a pattern may be used more advantageously than a finger-like one. Then, $\mathbf{M}^{-1}$ is used as the preconditioner to the linear system (11) solved using GMRES method.

This preconditioner have been demonstrated to be effective for matrices that are smooth with non-smooth diagonal bands only [10]. However, as our problem is smooth, but with both diagonal and off-diagonal non-smooth bands, see Fig. 1. It will no longer be possible to find a suitable bandwidth to include all of the required entries of $\hat{\mathbf{A}}$ and thus the DWTPer preconditioner will not be a close approximation to $\hat{\mathbf{A}}$.

A better preconditioner, suitable for our problem, is the modified DWTPer preconditioner found in [12] that will include all large entries corresponding to the non-smooth bands and that corresponding to the weighted averages. The DWTPer preconditioner is further improved in [11] and [24]. This is denoted by DWTPerMod preconditioner. In constructing this preconditioner, any non-smooth areas are identified and the matrix is permuted to arrange any localized non-smooth off-diagonal bands to the bottom or the right-hand edges of the matrix to form a bordered block matrix. Then, a DWTPer is applied to the permuted system. The resulting transformed matrix $\hat{\mathbf{A}}$ has an arrow-shaped sparsity pattern above some threshold with predicted increase in bandwidths [12]. This pattern suffers from little fill-in during LU factorization step.

Table 1. Number of iterations of GMRES method with to $l = 10^{-6}$

N	32 ($ka = 5$)	64 ($ka = 10$)	128 ($ka = 10$)	256 ($ka = 20$)
P1	11	15	14	20
P2	11	15	15	25
P3	7	14	10	17
P4	6	6	9	17
P5	7	9	4	10

Table 2. CPU-time in ms of GMRES method

N	32	64	128	256
P2	5.17053	7.86619	5.24753	7.31470
P5	4.02172	1.78506	1.41279	1.48344

Table 3. % Normalized error in computation

N	32	64	128	256
P1	0.03200	0.07900	0.04700	0.14100
P2	0.0310	0.01500	0.01700	0.10900
P3	0.0300	0.11000	0.04700	0.15600
P4	0.00100	0.07900	0.09300	0.12500
P5	0.00100	0.01500	0.01600	0.06300

Further, in [24] tighter bounds for the bandwidths are given and an optimal level l of DWTPer is determined, inherently, based on the minimization of the number of nonzero entries.

Based on the modifications to the DWTPer proposed in [12] and [24] we constructed the DWTPer-Mod preconditioner and found it to be effective in that it reduces both fill-in of LU factorization and the number of iterations required for convergence to a required tolerance. Accordingly, both computational complexity and convergence rate are improved.

The cost associated with the construction of the preconditioners is mainly in the application of the wavelet transform. Other costs related to permutation of the rows and columns, or thresholding and cutting are not expected to have a significant value. Thus, the cost in terms of flops of performing a standard DWT, and similarly that of DWTPer and DWT-PerMod, order m, level l to an $n \times n$ matrix is [12] $8mn^2(1 - 1/2^l)$.

If the preconditioner is effective, this additional cost in constructing the preconditioner could be acceptable due to the reduction in the number of GMRES iterations required for convergence to the required tolerance.

5. RESULTS

The integral equation formulation of the acoustic scattering problem as defined in (1) is solved using the proposed discretization scheme considering different M divisions on the surface of an acoustically hard scatterer.

The incident field is taken as a plane wave and the field frequency is taken over a wide frequency range covering the range of $ka = 1...20$ while k is the propagation constant and a is the scatterer characteristic length.

The results obtained are, then, compared based on a normalized error from analytical solution. The normalized error is defined as the ratio between the field (ϕ) error to the analytical solution as follows

$$\text{Normalized error} = \frac{\|\phi_{wvl} - \phi_{ana}\|}{\|\phi_{ana}\|}, \qquad (16)$$

where wvl is the computed numerical solution using the studied methods, ana is the analytical solution given in [25] for a hard acoustically sphere, and $\|.\|$ is the l_2 norm.

On the other hand, the integral equation of acoustic scattering in (1) is solved for a different wavenumbers k and sphere radii a with an incident plane wave on an acoustically hard sphere for obtaining ϕ_{wvl}. We, then, illustrate the effectiveness of the presented preconditioners through comparing the performance of GMRES using the following methods

- P1 Diagonal preconditioner.
- P2 DWT matrix without preconditioning.
- P3 Standard DWT Preconditioner.
- P4 DWTPer Preconditioner.
- P5 DWTPerMod Preconditioner.

The choices of wavelet family are large and the matrix dimension is dependent on the wavenumber and should be in the order of integer power of two,

Table 4. Number of Iterations of GMRES method for $N = 128$ with different values of ka

ka	10	11.12	12.24	13.36	14.48	15	15.6	16.72	17.84	18.96
P2	15	15	15	16	16	17	17	19	20	20
P5	4	5	5	6	7	7	8	9	10	10

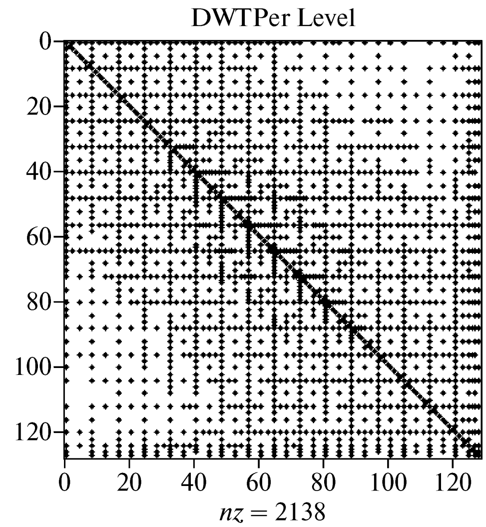

Fig. 3. DWTPer Preconditioner, $N = 128$.

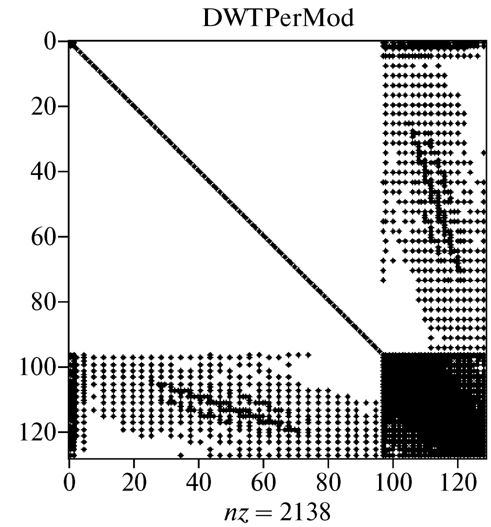

Fig. 4. DWTPerMod Preconditioner, $N = 128$.

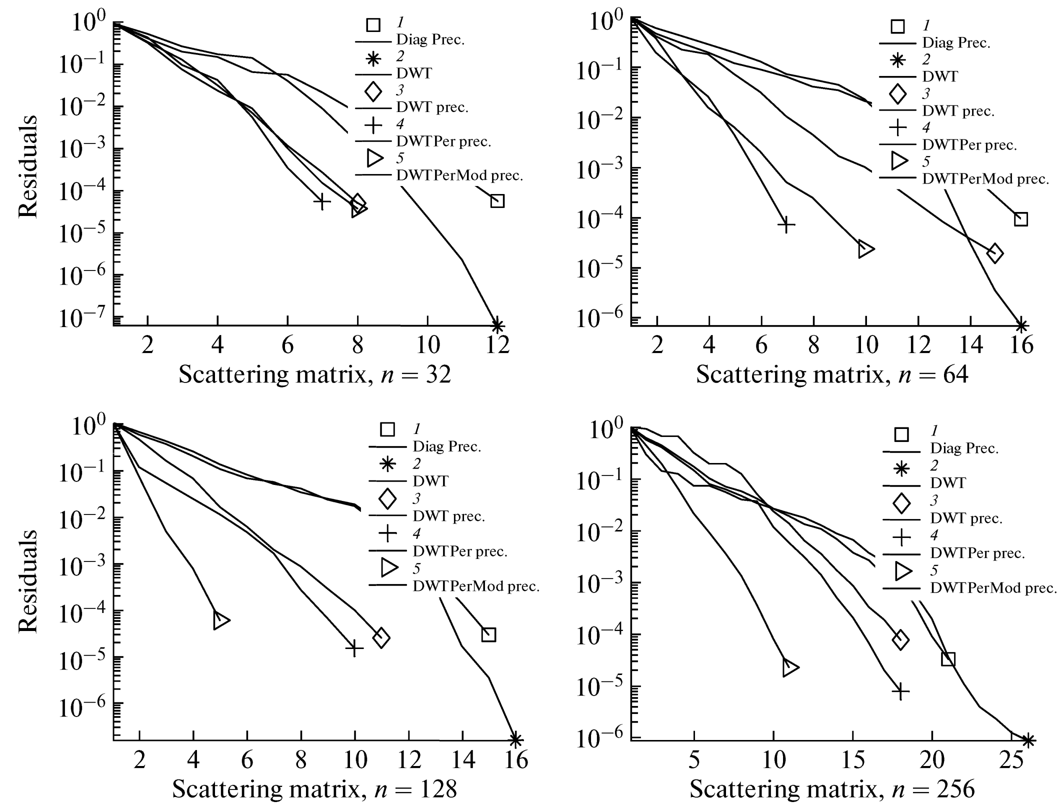

Fig. 5. Convergence behavior of P1–P5 methods.

i.e. (2^n). So, we confine the dimension to selective values as indicated in Table 1. In each case, the normalized wavenumbers, *ka*, are taken in a wide range between 1...20 and the wavelet family is taken as Daubechies of order 4. In each case above using GMRES, we restarted GMRES after 20 iteration steps and stopped the iteration when the relative residual norm fell below 10^{-6}.

Firstly, Table 1 lists the number of iteration steps required by the GMRES method to converge to the required tolerance, for the wavenumber *ka* and problem size *N*. The problem size is taken to be proportional to the wavenumber [26]. Secondly, the corresponding CPU-time for each case is given, in Table 2. In Table 3, the normalized error as defined in (16) is presented for P2 and P5 where one could

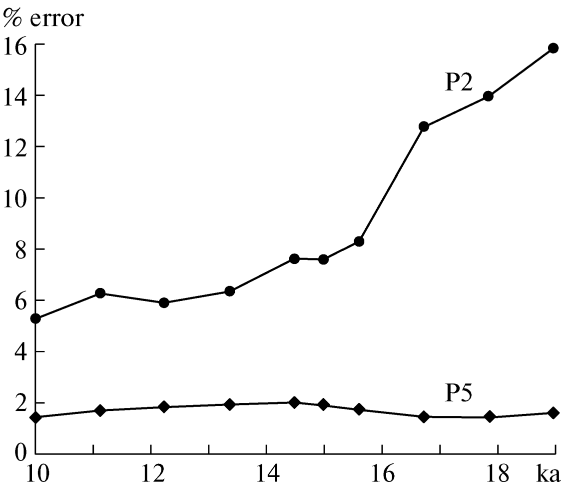

Fig. 6. Normalized error versus wavenumber ka for DWT and DWTperMod preconditioners for $N = 128$.

observe the low accuracy of computing the solution directly from the transformed system which could be acceptable for some applications requiring less computational time. Figures 2–4 show the sparsity pattern of a level $L = 3$ and $N = 256$ for the DWT, DWTPer, and the DWTPerMod preconditioners. In addition, the residual rate of convergence is given in Figure 5 to show the convergence behavior of the different methods.

As mentioned in the previous section the construction of the DWT-based preconditioners encounters an amount of overhead. This is compensated by the reduction in the number of iterations especially for higher dimensional problems, as illustrated in Table 1. The DWTPerMod preconditioner outperforms the other preconditioners with the number of iterations for high dimensions and less number of operations per iteration step due to the structure of preconditioner, as shown in Fig. 4. As mentioned in Section 4, an optimum level of transform is, inherently, computed to reduce the number of nonzero entries of the transformed system matrix. In general, the number of iterations using GMRES is reduced using DWT-based preconditioner methods.

Another point of comparison is obtained by fixing the problem size to a sufficiently large number ($N =$ 128) and comparing the performance of different solution methods based on the number of iterations and solution accuracy. In Table 4, the number of iteration steps of the GMRES method is depicted for each solution method at different normalized wavenumber (ka) with fixed number of discretization points at 128. For this case, the corresponding normalized error is presented in Fig. 6.

6. CONCLUSIONS

The dense matrix of an integral equation is sparsified and solved using DWT. A typical case of acoustic scattering is considered for illustrative purposes. Different sparsificatoin approaches are considered using DWT. The results show that DWT-based preconditioners are efficient for acoustic scattering problem. This is due to the smoothness property of its coefficient matrix. It has been shown that further study of the matrix structure could greatly improve the computational cost using an iterative solver. Several DWT-based preconditioners were implemented and have shown to outperform the diagonal preconditioning.

The use of the thresholded DWT matrix has shown to improve the computational time as dealing with a sparse matrix is solved more efficiently than a dense matrix. This improvement is on the expense of system accuracy, which is acceptable in some applications. DWT-based preconditioners achieve higher accuracy on a slight increase in the computational load of constructing the preconditioner.

Different cases of acoustic scattering on a hard acoustical sphere are studied on a wide range of the normalized wavenumber $ka = 1...20$ and the wavelet family is taken as Daubechies of order 4. GMRES iterative method is used in solving the resulting system of equations for acoustic scattering problem and compared to direct DWT solution and standard diagonal preconditioner.

The study shows that DWTPerMod Preconditioner and standard DWT give higher accuracy than the solution obtained directly from the transformed system. The sparsity pattern improves in the case of employing DWTPerMod Preconditioner.

The construction of the DWT-based preconditioners encounters an amount of overhead. This is compensated by the reduction in the number of iterations especially for higher dimensional problems. The DWTPerMod preconditioner outperforms the other preconditioners with the number of iterations for high dimensions and less number of operations per iteration step due to the structure of preconditioner. In general, the number of iterations using GMRES is reduced using DWT-based preconditioner methods.

Another point of comparison is obtained by fixing the problem size to a sufficiently large number ($N =$ 128) and comparing the performance of different solution methods based on the number of iterations and solution accuracy. In this case, the DWTPerMod preconditioner outperforms the other preconditioners.

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